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OPTIMIZATION OF A MULTIOBJECTIVE PROGRAMMING PROBLEM

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ABSTRACT

The purpose of this paper is to suggest the use of the Generalized Distance Approach in the field of multiobjective programming to find "compromising" conditions on the decision variables that will give a "near" optimal value for each of the objectives.

A global optimization technique called the Controlled Random Search (CRS) procedure has been used to minimize the distance function derived from the Generalized Distance Approach. It is numerically shown that this approach works well in solving multiobjective optimization problems.

ABSTRAK

Tujuan kertas ini adalah untuk mencadangkan penggunaan Pendekatan Jarak Teram dalam bidang Pengaturcaraan Berbilang Objektif untuk mencari keadaan-keadaan kompromi pada pembolehubah-pembolehubah keputusan yang boleh memberikan nilai hampir optimum bagi setiap objektif yang ingin dicapai.

Satu teknik pengoptimuman secara global iaitu prosidur Carian Rawak Terkawal telah digunakan untuk meminimumkan fungsi jarak yang diterbit dari Pendekatan Jarak Teram. Adalah ditunjukkan secara berangka bahawa pendekatan ini boleh digunakan untuk menyelesaikan masalah-masalah pengoptimuman berbilang objektif.

INTRODUCTION

The field of multiobjective programming dates back to the early 1950s [see Ignizio (1982), and Charnes and Cooper (1977)]. It is an approach used for the analysis of decisions that are surrounded by multiple objectives subjected to several constraints. The constraints can be in the form of limited resources or restrictive guidelines. Multiobjective programming has been applied for solving problems arising in such areas as production planning, transportation system planning and routing, airline crew scheduling, air pollution control, traffic signal planning, police patrols, water quality control, forest management and harvest policies, and many others.

Multiple objectives often conflict with each other. For example in an investment problem, one may simultaneously want to maximize the total expected return, minimize the amount of risk involved, and minimize the tax liability. Optimizing one objective may result in worsening at least one other objective. Cases where all the objectives reach their own optimum with the same solution point are extremely rare. Heuristically, one may superimpose the graphs of all the objectives and try to obtain the point(s) that would produce the nearest to the optimal result for all the objectives. Though this step is simple, it still has its limitations especially when the system is large and consists of several input variables (decision variables) and objectives. Furthermore, it is difficult to trace the optimal point(s) through this method.

The graphical approach described above is only one of the many ways to evaluate these trade-offs. Other existing approaches to solving multiobjective programming problems include the following:

- **Weighting or utility method**
  The weighting or utility method applies a set of weights to the objective function in order to add the functions to form one aggregate function, i.e., transform a multiple objective model into a single-objective model. The single function can then be solved to yield an efficient solution.

- **Ranking or prioritizing method**
  Objectives are ranked according to the importance perceived by the decision-maker. The decision-maker wishes to determine the feasible solution that would achieve, or most nearly achieve, the ranked objectives.

- **Efficient solution (or generating) method**
  The efficient solution method generates a total set of all the efficient solutions (or nondominated solutions), and from these various feasible solutions, the decision-makers are allowed to select the solution they subjectively prefer.

For a more detailed explanation on the above approaches, refer to Ignizio (1982), Mathur and Solow (1994), and Eppen *et al.* (1998).

TERMINOLOGY

Variables – usually denoted as X_i (i = 1, 2, 3, ... , n), is a factor subject to change within a problem. That is, its value may change, or at least change within certain limits.

Decision input variables – variables that are under the control of the decision-maker and could have an impact on the solution to the problem of interest.

Objective function – the overall objective of a decision problem represented by mathematical functions of the decision variables. Objective functions may also be linear or nonlinear in form, although herein we concentrate primarily on those of a linear form.

Constraint – a restriction on the values of variables in a mathematical model typically imposed by outside limitations.

Feasible solution region – the area in which all the points satisfy all the constraints simultaneously.

OBJECTIVE

In the field of multiobjective programming, the problem of optimization becomes more complicated when the number of objectives being analyzed is simultaneously increased. The problems arise because very often these objectives conflict with one another. Improving the value of one objective may result in worsening of one or more of the other objectives. As the simultaneous optimization of objectives is almost impossible, one may need to search for a "compromising" condition on the decision variables under which each objective deviates as little as possible from the ideal optimal value, that is, under such conditions, we have a so-called "near optimum" for each of the objectives.

In this research, a technique known as the Generalized Distance Approach is suggested for the study. This approach has been used in the field of Response Surface Methodology to find "compromising" conditions for the input variables that will give a "near" optimal value for each of the responses [see Khuri and Cornell (1987)]. Based on this approach, the multiobjective optimization problem involves the determination of the values of the decision variables that will minimize a certain distance function within a region (feasible solution region) of interest. However, often, calculation by hand is impossible owing to the complicated form of the distance function. As such, the aid of a computer is necessary. For the purpose of this research, a global optimization technique called the Controlled Random Search (CRS) procedure invented by W. L. Price in 1977 has been programmed and used to minimize the distance function. The CRS flow chart is given in appendix A. For a more detailed understanding of the CRS procedure, one may refer to Price (1977), Dixon and Szegö (1978), Conlon (1985 & 1992) and Lim (1993).

METHODOLOGY

Multiobjective Optimization Using The Generalized Distance Approach

We assume that all objective functions in a multiobjective system depend on the same set of decision variables, $X_1, X_2, \ldots, X_n$. If there are k objectives of interest and m linear constraints, then under the assumption mentioned above, the multiobjective programming problem could be written as:

Maximize (Minimize):

$$Z_1 = \sum_{j=1}^{n} C_{1j} X_j$$

$$Z_2 = \sum_{j=1}^{n} C_{2j} X_j$$

$$\vdots \qquad \textit{(objective functions)}$$

$$Z_k = \sum_{j=1}^{n} C_{kj} X_j$$

$$\sum_{j=1}^{n} A_{1j} X_j \leq b_1$$

$$\sum_{j=1}^{n} A_{2j} X_j \leq b_2$$

Subject to:

$$\vdots \qquad \textit{(constraints)}$$

$$\sum_{j=1}^{n} A_{mj} X_j \leq b_m$$

$$X_j \geq 0, \ j = 1, 2, \ldots, n \qquad \textit{(nonnegativity constraints)}$$

where $C_{1j}, C_{2j}, \ldots, C_{kj}, A_{1j}, A_{2j}, \ldots, A_{mj}, b_1, b_2, \ldots, b_m$ are given coefficients. In matrix notation, this problem can be written as:

Maximize (Minimize):

$$Z = CX \tag{1}$$

Subject to:

$$AX \leq b$$

$$X \geq 0$$

where Z is a [k x 1] vector, C is a [k x n] matrix, X is a [n x 1] vector, A is a [m x n] matrix and b is a [m x 1] vector.

Let the optimal value of Z_i optimized individually over the feasible solution region (i = 1, 2, ... , k) be ϕ_i and let $\phi = (\phi_1, \phi_2, \ldots, \phi_k)'$. If these individual optimal values, ϕ_i's, occur at the same point, X, then an "ideal" optimum is said to have been achieved. In this case, no further work is needed since the problem of multiobjective optimization is obviously solved. However, an ideal optimal value rarely exists in practice. Hence, in a multiobjective optimization problem, one possible approach is to search for "compromising" conditions on the decision variables which result in a "near" optimal value for each of the objective functions.

In Response Surface Analysis, such a deviation in the "compromising" conditions from the ideal can be formulated by means of a distance function which measures the distance from ϕ, the vector of individual optimal values, to Z, considered as a point in the feasible solution region. This method is suggested as an alternative approach to solve the optimization problems in multiobjective programming.

Let us denote this distance function by $\rho[Z, \phi]$. The multiobjective optimization problem becomes one of searching for conditions on X that minimize $\rho[Z, \phi]$ over the feasible solution region. There are many ways in which the distance function ρ can be chosen [See Khuri and Cornell (1987), pp. 290]. A commonly used distance function particularly for those who like to consider relatives changes from the individual optima, is:

$$\rho[Z, \phi] = \left[\sum_{i=1}^{k} \frac{(Z_i - \phi_i)^2}{\phi_i^2} \right]^{1/2} \qquad (2)$$

The distance function above is regarded as a generalized distance. It is simply a measure of the distance between Z and ϕ after it has been properly scaled. Suppose X_0 is the point in the feasible solution region at which $\rho\,[Z, \phi]$ achieves its absolute minimum. If m_0 is the minimum value of the distance function, then at X_0, all the objective functions can be described as being near optimal. It is always desirable to have small value of m_0 because the smaller the value of m_0, the closer the "compromising" conditions are to the conditions where each objective attains its individual optimum, respectively.

In summary, the steps to be taken to optimize a multiobjective function consisting of k objectives are as follows:

1. Optimize the objectives individually over the feasible solution region, as defined by the constraints, to obtain the vector f of individual optimal values. This can be done using linear mathematical programming techniques.
2. A distance measure, ρ, is chosen. For example, using Equation (2).
3. The distance function $\rho\,[Z, \phi]$ chosen in step 2 is minimized over the feasible solution region. Here, the CRS procedure can be applied effectively.

NUMERICAL EXAMPLES

Example 1 (Product Mix Example)
The example below is from Cook and Russell (1993).

The Faze Linear Company is a small manufacturer of high-fidelity components for the discriminating audiophile. It currently manufactures power amplifiers (amps) and preamplifiers (preamps); it has the facilities to produce only power amps, only preamps, or a combination of both. Production resources are limited, and it is critical that the firm produce the appropriate number of power amps and/or preamps in order to maximize profit. Each power amp contributes \$300 to profit and each preamp contributes \$500. The objective function representing short-term profit can be expressed as:

Maximize $Z_1 = 300X_1 + 500X_2$
where X_1 refers to the number of power amps to be produced each day, and X_2 is the number of preamps to be produced each day.

Suppose further that Faze Linear is planning to introduce a compact digital disc player that will require the Faze Linear preamp to operate it. Since Faze Linear expects compact disc players to be a significant product in the current decade, they would like to sell as many of their preamps as possible to accommodate the future introduction of the disc player. In terms of long range profits the company estimates that preamps will contribute four times as much profit as power amps. Thus, their long-term profit function can be expressed as:

$$\text{Maximize } Z_2 = X_1 + 4X_2$$

In this problem, the production process is limited by scarcity of high-quality transistors for the preamps, assembly worker hours, and inspection and testing worker hours. Because of a shortage of high-quality transistors, at most 40 preamps can be manufactured on a daily basis. There are only 240 hours of assembly worker time available each day. Furthermore, each power amp requires 1.2 hours for assembly and each preamp requires 4 hours. Finally, there are 81 worker hours available for inspection and testing each day, and the two components require 0.5 and 1 hour, respectively.

Based on the above information, the complete multiple objective formulation of the Faze Linear product mix problem can be expressed as shown below:

$$\begin{aligned}
&\text{Maximize } Z_1 = 300X_1 + 500X_2 \\
&\text{Maximize } Z_2 = X_1 + 4X_2 \\
&\text{Subject to:} \\
&\qquad X_2 \leq 40 && (3) \\
&\qquad 1.2X_1 + 4X_2 \leq 240 && (4) \\
&\qquad 0.5X_1 + X_2 \leq 81 && (5) \\
&\qquad X_1, X_2 \geq 0
\end{aligned}$$

We now consider the techniques used to determine the optimal value for each of the decision variables.

(i) Graphical Approach

Figures 1 and 2 are the graphs for Z_1 and Z_2 respectively which are plotted using the same scales with X_2 versus X_1.

Figure 1
Optimal Solution for Z_1

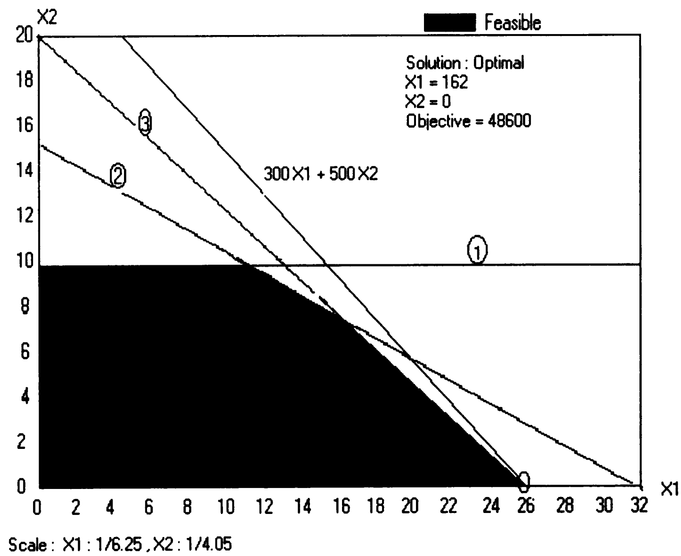
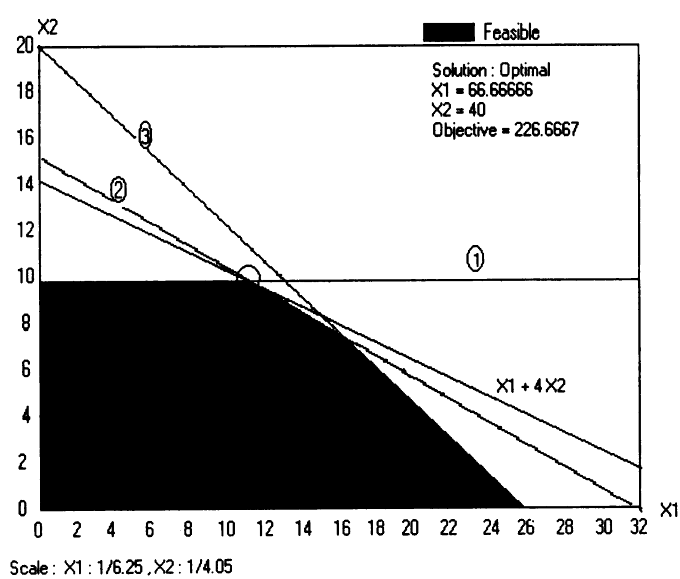


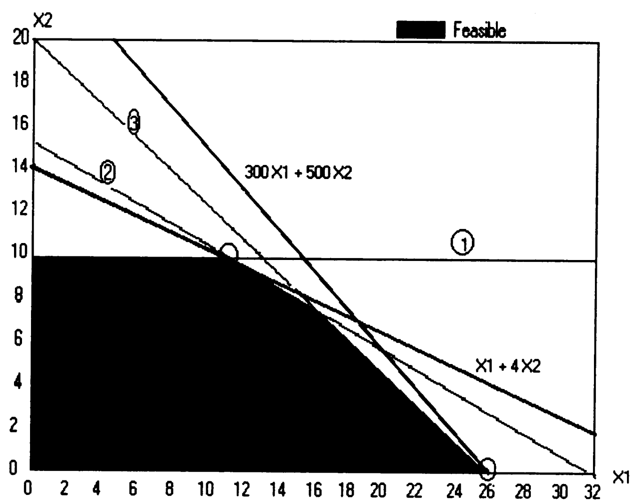
Figure 2
Optimal Solution for Z_2



From the figures, if we consider only the first objective function, its value Z_1 is maximized at the point (162, 0). The second objective function's value Z_2 is maximized at the point (66.7, 40).

The result of superimposing the two graphs is shown in Figure 3.

Figure 3
The Result of Superimposing Z_1 and Z_2



Obviously, it is difficult to identify one set of conditions (or one point in the feasible solution region) as being optimal with such a procedure. The only points that are not dominated are the points along the boundary from point (66.7, 40) to point (162, 0). Cook and Russell (1993), have summarized the extreme point solutions in the set of nondominated solutions, which and are shown in Table 1.

Table 1
Nondominated Extreme Points

Point	Objective Z_1	Objective Z_2
(66.7,40)	40,000	226.7
(105,28.5)	45,750	219.0
(162,0)	48,600	162.0

The choice of the best solution would depend on the relative trade-offs between the advantages and disadvantages of each of the three nondominated extreme points.

(ii) Generalized Distance Approach

From Figure 1 and 2, we know that the individual optimal values for Z_1 and Z_2 are 48,600 and 226.7, respectively. If we substitute these values into Equation (2), then the distance function ρ to be minimized is as follows:

$$\rho\,[Z, \phi\,] = \left[\frac{(Z_1 - 48{,}600)^2}{48{,}600^2} + \frac{(Z_2 - 226.7)^2}{226.7^2}\right]^{1/2}$$

$$= \left[\frac{(300X_1 + 500X_2 - 48{,}600)^2}{48{,}600^2} + \frac{(X_1 + 4X_2 - 226.7)^2}{226.7^2}\right]^{1/2}$$

Using the CRS procedure, the set of conditions that is "near" optimum for the objectives is obtained at $X_1 = 105$ and $X_2 = 28.5$. At this point, $Z_1 = 45{,}750$ and $Z_2 = 219$, respectively. This solution coincides with the solution (one of the nondominated extreme points) obtained using the Graphical Approach.

Example 2 (A Multiobjective Production Model)

[This example is taken from the book "Continuous Optimization Models", by Eiselt, Pederzoli and Sandblom (1987)].

A company has the facilities to produce two camera models: an economy and a luxury model. Each of these two models can either be sold in the domestic market or can be exported. The following table (Table 2) indicates the labour and machine requirements, the domestic and export selling prices, as well as resource availabilities and resource costs:

Table 2

Data for the Two Camera Models

	Labour [hours/quantity unit]	Machine [quantity units/ hour	Selling price	
			Domestic	Export
Economy model	22	1	$1,000	$ 800
Luxury model	35	1/2	$1,740	$1,020
Resource availabilities (hours)	1,620	81		
Costs [$/hour]	30	20		

The company has three relevant objectives:

(1) maximize profit
(2) minimize layoffs
(3) maximize the number of exported cameras

where the last objective is part of a promotional campaign to establish a reputation abroad.

In order to formulate the complete multiple objective model for the above situation, we define X_1 and X_2 as the number of economy models for the domestic and export market respectively, and X_3 and X_4 as the number of luxury models produced for (and sold in) the domestic and export market, respectively. Then the multiple objective model is:

$$\text{Maximize } Z_1 = 320X_1 + 120X_2 + 650X_3 - 70X_4 \quad \text{(profit)}$$
$$\text{Maximize } Z_2 = 22X_1 + 22X_2 + 35X_3 + 35X_4 - 1620 \quad \text{(layoffs)}$$
$$\text{Maximize } Z_3 = X_2 + X_4 \quad \text{(exported cameras)}$$

Subject to:

$$22X_1 + 22X_2 + 35\,X_3 + 35X_4 \leq \;\; 1620 \qquad \text{(labour)}$$
$$X_1 + X_2 + 2X_3 + 2X_4 \leq 81 \qquad \text{(machine)}$$
$$X_1\,,\, X_2\,,\, X_3\,,\, X_4 \geq 0$$

For this problem, it is stated in Eiselt, Pederzoli and Sandblom (1987) that there are four noninferior solutions, namely:

$X^1 = [45, 0, 18, 0]'$ with $Z(X^1) = [26{,}100; 0; 0]'$
$X^2 = [0, 45, 18, 0]'$ with $Z(X^2) = [17{,}100; 0; 45]'$
$X^3 = [0, 73\ 7/11, 0, 0]'$ with $Z(X^3) = [8{,}836\ 4/11; 0; 73\ 7/11\ ]'$
$X^4 = [0, 0, 40\ 1/2, 0]'$ with $Z(X^4) = [26{,}325; -202\ 1/2, 0]'$

where X^4 maximizes the profit; X^1, X^2 and X^3 all minimize layoffs and X^3 maximizes exports. All noninferior solutions, however, indicate that no luxury models should be produced for market abroad.

We now consider the use of the Generalized Distance Approach to solve this multiobjective production problem. From the above statement, we know that, in this case, the vector ϕ of individual optimal values is (26,325, 0, 73 7/11). If we substitute these values into Equation (2) and minimize $\rho[Z, \phi]$ over the feasible solution region using the CRS procedure, the "compromising" conditions on the decision variables under which the multiobjective function deviates as little as possible from the ideal optimal value is obtained at X = [0.0, 49.73, 15.0, 0.0]' with Z = [15,736.16; 0.0; 49.73]'. Again this solution also indicates that no luxury models should be produced for markets abroad.

CONCLUSION

In this research, it is numerically shown that the Generalized Distance Approach can be used as an alternative approach to solving multiobjective programming problems. One advantage in using the Generalized Distance Approach is that it can be used in a large system involving several decision variables (> 2) where the display of a graph is impossible. However, further research is needed to compare this approach with some other available approaches in order to popularize the use of Generalized Distance Approach in the field of multiobjective programming.

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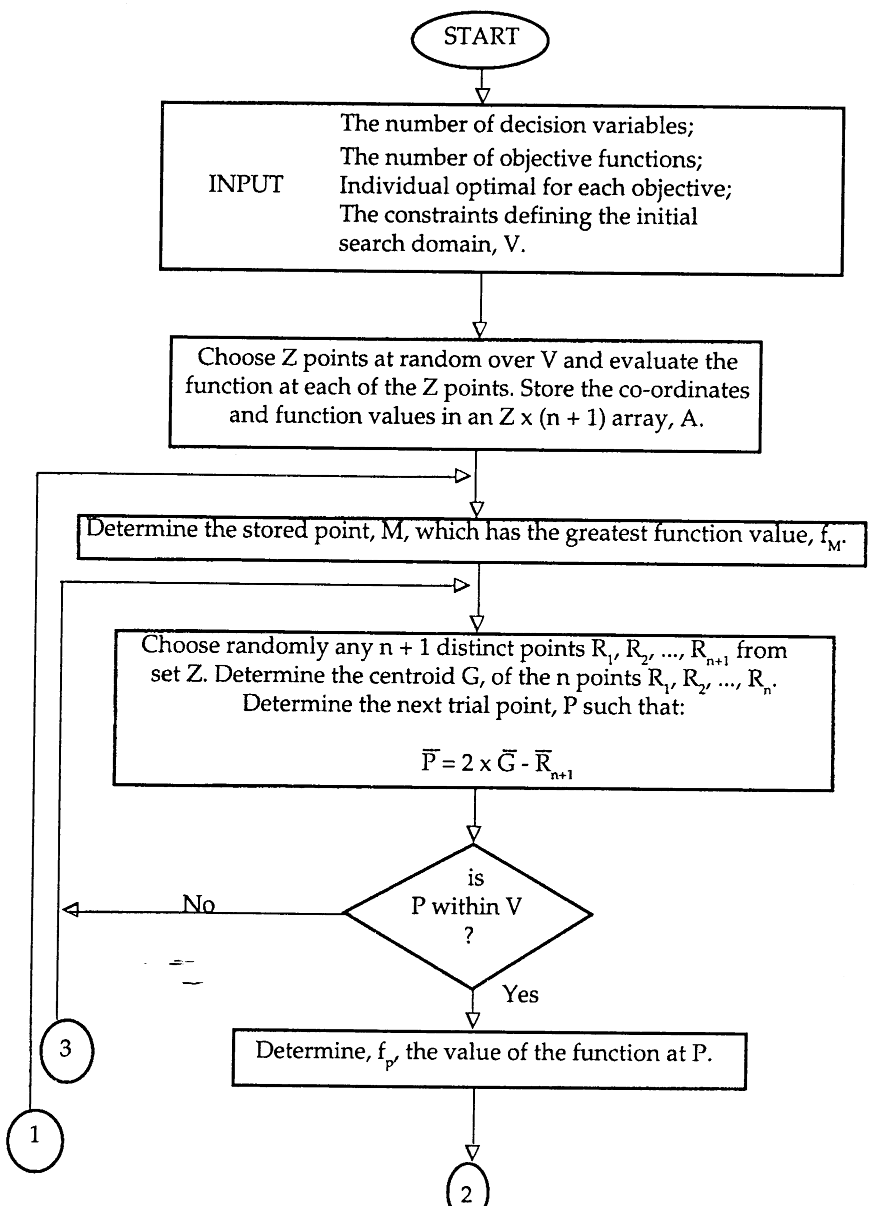
APPENDIX A

CONTROLLED RANDOM SEARCH FLOW CHART



(Appendix A - continued)